NUMERICAL SOLUTION OF TRANSFER THEORY PROBLEMS BY THE DIRECT REDUCTION METHOD

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A reduction method is presented for solution of stationary and nonstationary problems in transfer theory for boundary conditions of the first and second sort.

We will consider a nonstationary heat-mass-transfer equation within the rectangular region $0 \le d \le d$, $0 \le y \le b$ for $0 < t \le T$:

$$\gamma \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \varkappa u = g(x, y, t);\ \gamma > 0,\ \varkappa \geqslant 0 \tag{1}$$

given initial conditions

$$u(x, y, 0) = f(x, y) \tag{2}$$

and boundary conditions of the first sort

$$\begin{aligned} u(0, y, t) = T_{(1)}(y, t),\ u(a, y, t) = T_{(3)}(y, t), \\ u(x, 0, t) = T_{(2)}(x, t),\ u(x, b, t) = T_{(4)}(x, t) \end{aligned} \tag{3}$$

or of the second sort

$$\begin{aligned} \frac{\partial u(x, 0, t)}{\partial x} = \varphi_{(1)}(y, t),\ \frac{\partial u(a, y, t)}{\partial x} = \varphi_{(3)}(y, t), \\ \frac{\partial u(x, 0, t)}{\partial y} = \varphi_{(2)}(x, t),\ \frac{\partial u(x, b, t)}{\partial y} = \varphi_{(4)}(x, t). \end{aligned} \tag{4}$$

In the stationary case Eq. (1) transforms to the Helmholz equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \varkappa u = g_{(1)}(x, y) \tag{5}$$

with boundary conditions of the first type

$$\begin{aligned} u(0, y) = T_{(1)}(y),\ u(a, y) = T_{(3)}(y), \\ u(x, 0) = T_{(2)}(x),\ u(x, b) = T_{(4)}(x) \end{aligned} \tag{6}$$

or second type

$$\begin{aligned} \frac{\partial u(0, y)}{\partial x} = \varphi_{(1)}(y),\ \frac{\partial u(a, y)}{\partial x} = \varphi_{(3)}(y), \\ \frac{\partial u(x, 0)}{\partial y} = \varphi_{(3)}(x),\ \frac{\partial u(x, b)}{\partial y} = \varphi_{(4)}(x). \end{aligned} \tag{7}$$

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Analytic solution of these problems can be carried to completion by the methods of separation of variables, finite integral transformations, Green functions, etc.

Such solutions, applicable over a wide range of parameter variation, in the form of functions q(x, y, t), f(x, y), T_i, φ_i (i = 1, 2, 3, 4) usually require the calculation of series, eigenfunctions, and various integrals to obtain numerical values, and thus even in the presence of a formal solution the determination of numerical values is a cumbersome problem.

Numerical solution of such problems can be performed by various methods — the grid method, variation, iteration, and direct methods, etc. One of the direct methods is Buneman's reduction method [1], which is called the direct reduction method [2, 3] or the decomposition method [4] in the literature.*

We will describe the essence of the method for the nonstationary case. We will denote the grid points by

$$z_{ij}^n = (ih_1;\ jh_2,\ n\Delta t),$$

where

$$h_1 = a/N_1,\ h_2 = b/N_2,\ \Delta t = T/M.$$

For boundary conditions of the first sort, the method requires $N_1 = 2^{S_1} + 2$ and $N_2 = 2^{S_2} + 2$, while for conditions of the second sort $N_1 = 2^{S_1} + 2$ and $N_2 = 2^{S_2} + 2$. We write the value of the grid function in the form

$$u_{ij}^n = u(z_{ij}^n).$$

We introduce a quantity with weight σ:

$$u_{ij}^{n(\sigma)} = \sigma u_{ij}^n + (1-\sigma)\, u_{ij}^{n-1}$$

and approximate Eq. (1) to an accuracy $0(h_1^2 + h_2^2 + \Delta t^2)$

$$\begin{aligned}&\frac{\gamma}{\Delta t}\,(u_{ij}^n - u_{ij}^{n-1}) - \frac{1}{h_1^2}\left(u_{i-1j}^{n\left(\frac{1}{2}\right)} - 2u_{ij}^{n\left(\frac{1}{2}\right)} + u_{i+1j}^{n\left(\frac{1}{2}\right)}\right)\\ &-\frac{1}{h_2^2}\left(u_{ij-1}^{n\left(\frac{1}{2}\right)} - 2u_{ij}^{n\left(\frac{1}{2}\right)} + u_{ij+1}^{n\left(\frac{1}{2}\right)}\right) + \varkappa u_{ij}^{n\left(\frac{1}{2}\right)} = g_{ij}^{n\left(\frac{1}{2}\right)},\\ &2 \leqslant i \leqslant N_1,\ 2 \leqslant j \leqslant N_2,\ 1 \leqslant n \leqslant M.\end{aligned} \tag{8}$$

Correspondingly, we will approximate conditions (2)-(4) thus:

$$u_{ij}^0 = f_{ij},\ u_{1j}^n = T_{(1)j}^n,\ u_{N_1+1j}^n = T_{(3)j}^n,\ u_{i,1}^n = T_{(2)i}^n,\ u_{iN_2+1}^n = T_{(4)i}^n, \tag{9}$$

$$\frac{1}{2h_1}\,(u_{3j}^n - u_{1j}^n) = \varphi_{(1)j},\ \frac{1}{2h_1}\,(u_{N_1+1j}^n - u_{N_1-1j}^n) = \varphi_{(3)j}^n, \tag{10}$$

$$\frac{1}{2h_2}\,(u_{i3}^n - u_{i1}^n) = \varphi_{(2)i},\ \frac{1}{2h_2}\,(u_{iN_2+1}^n - u_{iN_2-1}^n) = \varphi_{(4)i}^n. \tag{11}$$

Equations (8)-(11) allow calculations with a large time step. A shortcoming of this scheme in the case of boundary conditions of the second sort is that the reduction method requires an approximation, Eq. (11), and thus is nonconservative [3].

We will consider a layer fixed in time (emitting the index). We rewrite Eqs. (8), (10), (11) in the form of a matrix three-diagonal scheme in which all known quantities lie on the right-hand side. For boundary conditions of the first sort

$$-u_{j-1} + Au_j - u_{i+1} = g_j,\ u_i = \begin{pmatrix} u_{2j} \\ \cdot\ \cdot\ \cdot \\ u_{K_1 j} \end{pmatrix}. \tag{12}$$

* The present study was carried out in 1976-1977 while Pol' was in the USSR, and the results were obtained independently from [3, 4], which studies the authors became aware of after the present study was printed.

Boundary conditions of the second sort reduce to relationships of the form

$$Bu_2 - 2u_3 = g_2,$$

$$-u_{j-1} + Bu_j - u_{j+1} = g_j,\ u_j = \begin{pmatrix} u_{2j} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ u_{N_2 j} \end{pmatrix}, \tag{13}$$

$$-2u_{N_2-1} + Bu_{N_2} = g_{N_2}.$$

Let $\beta = \left(\frac{h_2}{h_1}\right)^2$ and $\alpha = 2(1+\beta) + h_2^2\left(\frac{\gamma}{\Delta t} + \varkappa\right)$. Then

$$A = \begin{pmatrix} \alpha & -\beta & & & \\ -\beta & \alpha & -\beta & & \\ & \cdot & \cdot & \cdot & \\ & & -\beta & \alpha & -\beta \\ & & & -\beta & \alpha \end{pmatrix}, \tag{14}$$

$$B = \begin{pmatrix} \alpha & -2\beta & & & \\ -\beta & \alpha & -\beta & & \\ & \cdot & \cdot & \cdot & \\ & & -\beta & \alpha & -\beta \\ & & & -2\beta & \alpha \end{pmatrix}. \tag{15}$$

The reduction, or Buneman [1], method uses doubly cyclic reduction. We use the notation

$$A^{(0)} = A,\ B^{(0)} = B,\ g_j^{(0)} = g_j.$$

The basic reduction formulas for boundary conditions of the first sort, Eq. (12), have the form

$$-u_{j-2^{t+1}} + A^{(t-1)}u_j - u_{j+2^{t+1}} = A^{(t)}g_j^{(t)} + g_{j+2^t}^{(t)} + g_{j-2^t}^{(t)} = g_j^{(t+1)},\ j \equiv 1\,(2^{t+1}), \tag{16}$$

$$A^{(t)+1} = [A^{(t)}]^2 - 2E,\ t = 0, \ldots, s_2 - 2,$$

while for boundary conditions of the second sort, Eq. (13),

$$B^{(t+1)}u_2 - 2u_{2+2^{t+1}} = B^{(t)}g_2^{(t)} + 2g_{2+2^t}^{(t)} = g_2^{(t+1)}, \tag{17}$$

$$-u_{j-2}^{t+1} + B^{(t+1)}u_j - u_{j+2^{t+1}} = B^{(t)}g_j^{(t)} + g_{j+2^t}^{(t)} + g_{j-2^t}^{(t)} = g_j^{(t+1)}, \tag{18}$$

$$-2u_{N_2-2^{t+1}} + B^{(t+1)}u_{N_2} = B^{(t)}g_{N_2}^{(t)} + 2g_{N_2-2^t}^{(t)} = g_{N_2}^{(t+1)},\ j \equiv 2\,(2^{t+1}), \tag{19}$$

$$B^{(t+1)} = [B^{(t)}]^2 - 2E,\ t = 0, \ldots, s_2 - 2.$$

For Eq. (18) it is necessary to continue the reduction

$$B^{(s_2)}u_2^{(s_2-1)} + 2 = B^{(s_2-1)}g_{2^{s_2-1}}^{(s_2-1)} + g_{N_2}^{(s_2-1)} + g_2^{(s_2-1)}, \tag{20}$$

$$B^{(s_2)} = [B^{(s_2-1)}]^2 - 4E. \tag{21}$$

In both cases the last reduction step permits obtaining a solution for the central vector, while Eqs. (16) and (18) allow calculation of all remaining vectors upon reduction of t.

In calculating the right-hand sides $q_j^{(t)}$ with Eqs. (16), (18), and (20) it is necessary to operate with a high order of numbers, (depending on dimensionality), which leads to an increase in computation error.

However, this shortcoming can be eliminated by recurrent formulas working with only inverse matrices [2]. Below we present a variant realizing the vectors $p_j^{(t)}$ and $q_j^{(t)}$ on a computer, for which

$$g_j^{(t)} = A^{(t)}p_j^{(t)} + q_j^{(t)}, \quad g_j^{(t)} = B^{(t)}p_j^{(t)} + q_j^{(t)}. \tag{22}$$

Let $p_j^{(0)} = 0$, $q_j^{(0)} = g_j^{(0)}$. Then for boundary conditions of the first sort

$$p_j^{(t+1)} = p_j^{(t)} + A^{(t)-1}(p_{j+2^t}^{(t)} + p_{j-2^t}^{(t)} + q_j^{(t)}), \quad q_j^{(t+1)} = q_{j+2^t}^{(t)} + q_{j-2^t}^{(t)} + 2p_j^{(t+1)}, \tag{23}$$

and for conditions of the second sort

$$\begin{gathered} p_2^{(t+1)} = p_2^{(t)} + B^{(t)-1}(2p_{2+2^t}^{(t)} + q_2^{(t)}), \quad q_2^{(t+1)} = 2(q_{2+2^t}^{(t)} + p^{(t+1)}), \\ p_j^{(t+1)} = p_j^{(t)} + B^{(t)-1}(p_{j+2^t}^{(t)} + p_{j-2^t}^{(t)} + q_j^{(t)}), \quad q_j^{(t+1)} = q_{j+2^t}^{(t)} + q_{j-2^t}^{(t)} + 2p_j^{(t+1)}, \\ p_{N_s}^{(t+1)} = p_{N_s}^{(t)} + B^{(t-1)}(2p_{N_s-2^t}^{(t)} + q_{N_s}^{(t)}), \quad q_{N_s}^{(t+1)} = 2(q_{N_s-2^t}^{(t)} + p_{N_s}^{(t+1)}). \end{gathered} \tag{24}$$

The last reduction step leads to

$$\begin{gathered} p_{2^{s_2}-1+2}^{(s_2)} = p_{2^{s_2-1}-1+2}^{(s_2-1)} \quad B^{(s_2-1)-1}(p_{N_s}^{(s_2-1)} + p_2^{(s_2-1)} + q_{2^{s_2-1}-1+2}^{(s_2-1)}), \\ q_{2^{s_2}-1+2}^{(s_2)} = q_{N_s}^{(s_2-1)} + q_2^{(s_2-1)} + 4p_{2^{s_2}-1+2}^{(s_2)}. \end{gathered} \tag{25}$$

The programs which were developed, like Buneman's original program [1], utilize the second variant, Eq. (22), which realizes only $q_j^{(t)}$. This method requires somewhat more computation time but is less demanding of machine memory, for which only the expressions

$$p_j^{(t+1)} = \frac{1}{2}(q_j^{(t+1)} - q_{j+2^t}^{(t)} - q_{j-2^t}^{(t)}), \quad p_{2^{s_2}-1+2}^{(s_2)} = \frac{1}{4}(q_{2^{s_2}-1}^{(s_2)} - q_{N_s}^{(s_2-1)} - q_2^{(s_2-1)}), \tag{26}$$

are required, eliminating the quantities $p_j^{(t)}$ in Eqs. (24) and (25) and leading to new recurrent formulas for $q_j^{(t)}$.

Revolution of the matrices $A^{(t)}$ and $B^{(t)}$, $t = 0, \ldots, s_2 - 1$ is accomplished with the aid of factored matrices

$$A^{(t)} = \beta^{2^t} \prod_{\nu=1}^{2^t} \left[\frac{1}{\beta} \left(A^{(0)} + 2\cos\left(\frac{2\nu - 1}{2^{t+1}} \pi \right) E \right) \right]. \tag{27}$$

The formula for $B^{(t)}$ is analogous. The factored B^{S_2} has the form

$$B^{(s_2)} = \beta^{2^{s_2}} \left(\prod_{\nu=1}^{2^{s_2-1}-1} \left[\frac{1}{\beta} \left(B^{(0)} + 2\cos\left(\frac{\nu}{2^t} \pi \right) E \right] \right)^2 \left[\frac{1}{\beta}(B^{(0)} + 2) \right] \left[\frac{1}{\beta}(B^{(0)} - 2) \right]. \tag{28}$$

Revolution of the matrices $A^{(t)}$, $B^{(t)}$, and $B^{(s_2)}$ leads to revolution of the three-diagonal matrices with the aid of reduction and multiplication by β^{-2t} or β^{-2s_2}. The form of the factorization formula is related to this. This method is more rapid than the drive method of [4], but leads to a limitation on N_1. In the case of $\gamma = \varkappa = 0$ the quantity $1/\beta\,(B^{(0)} - 2)$ will be irregular, but then the solution can be defined at a single point.

The reduction method was realized in the form of two programs for solving nonstationary and stationary equations for boundary conditions of the first and second sort. A number of test problems were solved with these programs on a Minsk-32 machine. For example, the problem of Eqs. (1)-(4) was solved with $\gamma = 1.205$, $g = 0$ and initial condition

$$f(x, y) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

at $g = 0$ and $T_1 = T_2 = T_3 = T_4 = 0$.

The theoretical and numerical solutions coincided to an accuracy of 10^{-4} at step values of $h_1 = 0.09$, $h_2 = 0.1$. Twelve points in time with a step $\Delta t = 0.1$ were calculated. Time required for calculating one variant was 5 min.

NOTATION

γ and $\varkappa$, coefficients; u, temperature, concentration, etc.; x and y, coordinates; t, time; Δ, Laplace operator; f(x, y), initial function; g(x, y, t) and T_1 (y, t); T_2(x, t); T_3(y, t); T_4(x, t); φ_1(y, t); φ_2(x, t); φ_3(y, t); φ_4(x, t), specified functions; s_1 and s_2, dimensionality coefficients of data blocks; $A^{(0)}$, $A^{(t)}$, $A^{(t-1)}$, matrices of algebraic system of equations for boundary conditions of first sort; $B^{(0)}$, $B^{(t)}$, B^{t-1}, matrices of system of equations for boundary conditions of second sort; g_i, vectors of right-hand sides; E, unit matrix; α, β, elements of three-diagonal matrix; μ_j, $\mu_{j-2}{}^{t+1}$, $u_{j+2}{}^{t+1}$, vectors of desired quantities; $q_i^{(t)}$, $p_i^{(t)}$, $t = 1, \ldots, s$ vectors in cyclical reduction; h_1 and h_2, step of space grid; Δt, step in time; a, b, dimensions of rectangle; N_1 and N_2, number of grid points along x and y axes, respectively; z^n_{ij}, grid; u^n_{ij}, grid function; σ, real parameter (weight).

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SOLUTION OF THE NONLINEAR INVERSE THERMAL CONDUCTIVITY PROBLEM BY THE ITERATION METHOD

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A regular iteration algorithm is constructed for the case of a nonlinear generalized thermal conductivity equation for determination of the nonstationary thermal flux. The algorithm is based on the method of conjugate gradients.

In experimental studies of nonstationary thermal processes, it becomes necessary to calculate thermal boundary conditions from temperature measurements within bodies (the inverse thermal conductivity boundary problem). The well-known incorrectness of the formulation of this inverse problem, which manifests itself as a strong sensitivity of the results to errors in the input information, requires the development of approximate algorithms which can suppress the instability of the results and maintain required accuracy.

We will consider the inverse problem for a nonlinear generalized thermal conductivity equation in the region $\{0 \le x \le b,\ 0 \le t \le t_m\}$. It is required that the dependence of thermal flux $q_1(t)$ on the left-hand boundary on the known temperature f(t) and the thermal flux $q_2(t)$ on the right-hand boundary be determined. Initial conditions are specified. Thus, we have

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + K(T)\frac{\partial T}{\partial x} + \varphi(T),\ 0 < x < b,\ 0 < t \leqslant t_m, \tag{1}$$

$$T(x, 0) = \xi(x), \tag{2}$$

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